Kuwait University	Math 102	June 1, 2004
Dept of Math & Comp Sci	Final Examination	Duration: 2 hours

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1. Given the points $P(3,-1,2)$, $Q(4,-2,3)$, and $R(1,-1,5)$.

   (a) Find parametric equations of the line passing through R and in the direction of $\overrightarrow{QP}$. (2 points)

   (b) Find the equation of the plane that contains all three points. (2 points)

   (c) Find the area of the triangle determined by P, R, and Q. (1 point)

2. Given $f(x) = 3x^2 - \sin^{-1}(2^x) + \frac{\pi}{6}$.

   (a) Find the domain of f. (1 point)

   (b) Show that f^{-1} exists, and find its domain. (2 points)

   (c) Why is the point $P(3,-1)$ on the graph of f^{-1}? Find the slope of the tangent line to the graph of f^{-1} at P. (2 points)

3. (a) Use logarithmic differentiation to find $y'(0)$ if

$$y = \frac{\sqrt[5]{x+1}\,\left(1+x^2\right)^x\, e^{\left(x^2+\tan^{-1}x\right)}}{\cosh x + \operatorname{sech} x}$$ (3 points)

   (b) Find $\lim_{x\to 0} |x|^{\sin x}$ (3 points)

4. Evaluate the following integrals. (3 points each)

   (a) $\int \sqrt{3+2x-x^2}\,dx$

   (b) $\int \frac{1}{\tan x + \sin x}\,dx$

   (c) $\int \frac{\sec^{-1} x}{x^2}\,dx$

   (d) $\int \frac{\tan x}{\sqrt{5\sec x - 4}}\,dx$

5. Does the integral $\int_0^\infty \frac{e^{-x}}{e^x+1}\,dx$ converge or diverge? If it converges, find its value. (4 points)

6. Find the length of the curve that has parametric equations

$$x = \tfrac{1}{5}t^5 \qquad y = \tfrac{1}{4}t^4 \qquad 0 \le t \le 1$$ (4 points)

7. Find the area inside the graphs of both polar equations $r = 1+\cos\theta$, and $r = -\cos\theta$. (4 points)

Q1

(a) A direction vector for the line is $u = \langle -1, 1, -1\rangle$. Parametric equations are $x = 1 - t,\quad y = -1 + t,$
$z = 5 - t \ (-\infty < t < \infty)$

(b) $v = \overrightarrow{QR} = \langle -3, 1, 2\rangle$. The vector $u \times v = \begin{vmatrix} i & j & k \\ -1 & 1 & -1 \\ -3 & 1 & 2 \end{vmatrix} = 3i + 5j + 2k$, is a normal to the required plane.

The equation of the plane is: $3(x-3) + 5(y+1) + 2(z-2) = 0$ or $3x + 5y + 2z - 8 = 0$.

(c) The area $= \frac{1}{2}\|u \times v\| = \frac{1}{2}\sqrt{(-3)^2 + 5^2 + 2^2} = \frac{1}{2}\sqrt{38}$

Q2

(a) Domain of $f = (-\infty, 0]$ since $f(x)$ is well-defined $\Leftrightarrow 0 < 2^x \leq 1 \Leftrightarrow x \leq 0$

(b) $f'(x) = 6x - (\ln 2)\frac{2^x}{\sqrt{1-2^{2x}}} < 0$ for $x < 0 \Rightarrow f$ is decreasing on $(-\infty, 0]$, so that f^{-1} exists.
$\lim\limits_{x\to-\infty} f(x) = \infty, \qquad f(0) = -\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$. Domain of $f^{-1} = [-\frac{\pi}{3}, \infty)$

(c) $f(-1) = 3 - \sin^{-1}\left(\frac{1}{2}\right) + \frac{\pi}{6} = 3 - \frac{\pi}{6} + \frac{\pi}{6} = 3 \Rightarrow f^{-1}(3) = -1$, so $P(3, -1)$ is on the graph of f^{-1}.
$\frac{df^{-1}}{dx}(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(-1)} = \frac{1}{-6-(\ln 2)\frac{2^{-1}}{\sqrt{1-2^{-2}}}} = -\frac{\sqrt{3}}{6\sqrt{3}+\ln 2}$

Q3

(a) $\ln|y| = \frac{1}{5}\ln|x+1| + x\ln(1+x^2) + x^2 + \tan^{-1}x - \ln(\cosh x + \operatorname{sech} x)$
$\frac{y'}{y} = \frac{1}{5}\frac{1}{x+1} + \frac{2x^2}{1+x^2} + \ln(1+x^2) + 2x + \frac{1}{1+x^2} - \dfrac{\sinh x - \tanh x \operatorname{sech} x}{\cosh x + \operatorname{sech} x}$
$y(0) = \frac{1}{2}$ so $y'(0) = \frac{1}{2}(\frac{1}{5} + 1) = \frac{3}{5}$

(b) Put $y = |x|^{\sin x}$, so $\ln y = (\sin x)\ln|x|$

$\lim\limits_{x\to 0}\ln y = \lim\limits_{x\to 0}(\sin x)\ln|x| = \lim\limits_{x\to 0}\frac{\ln|x|}{\csc x} = -\lim\limits_{x\to 0}\frac{\frac{1}{x}}{\csc x \cot x} = -\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)\tan x = -(1)(0) = 0$
$\lim\limits_{x\to 0}|x|^{\sin x} = \lim\limits_{x\to 0} e^{\ln y} = e^0 = 1$

Q4

(a) $\int\sqrt{3+2x-x^2}\,dx = \int\sqrt{4-(x-1)^2}\,dx \overset{(x-1=2\sin\theta)}{=} \int 4\cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)\,d\theta$
$= 2(\theta + \frac{1}{2}\sin 2\theta) + C = 2\sin^{-1}(\frac{x-1}{2}) + \frac{1}{2}(x-1)\sqrt{3+2x-x^2}) + C$

(b) $u = \tan\frac{x}{2}$, $dx = \frac{2du}{1+u^2}$, $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$
$\int\frac{1}{\tan x + \sin x}dx = \int\frac{1}{\left(\frac{2u}{1+u^2}\times\frac{1+u^2}{1-u^2} + \frac{2u}{1+u^2}\right)}\frac{2}{1+u^2}du = \int\frac{1-u^2}{2u}du = \frac{1}{2}\int(\frac{1}{u} - u)du$
$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C$

(c) $\int\frac{\sec^{-1}x}{x^2}dx \overset{(u=\sec^{-1}x \quad dv=\frac{1}{x^2}dx)}{=} -\int\left(\frac{1}{x}\right)'\sec^{-1}x\,dx = -\frac{1}{x}\sec^{-1}x + \int\left(\frac{1}{x}\right)\frac{1}{x\sqrt{x^2-1}}dx$ $(x = \sec\theta)$

$= -\frac{\sec^{-1}x}{x} + \int\overbrace{\frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}}^{\cos\theta}d\theta = -\frac{\sec^{-1}x}{x} + \sin\theta + C = -\frac{\sec^{-1}x}{x} + \frac{\sqrt{x^2-1}}{|x|} + C$

(d) $\int\frac{\tan x}{\sqrt{5\sec x - 4}}dx \underset{u=\sqrt{5\sec x-4}}{=} 2\int\frac{du}{u^2+4} = \tan^{-1}\frac{u}{2} + C = \tan^{-1}(\frac{1}{2}\sqrt{5\sec x - 4}) + C$

$= \sec^{-1}\left(\frac{\sqrt{5\sec x}}{2}\right) + C$

12. Q5 $\displaystyle\int \frac{e^{-x}}{e^x+1}\,dx = \int \frac{e^x\,dx}{e^{2x}(e^x+1)} \overset{(u=e^x)}{=} \int \frac{du}{u^2(u+1)} = \int \left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{u+1}\right) du = -\frac{1}{u} + \ln\frac{u+1}{u} = -e^{-x} + \ln(1+e^{-x})$

$$\int_0^\infty \frac{e^{-x}}{e^x+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^{-x}}{e^x+1}\,dx = 1 - \ln 2 + \lim_{t\to\infty}\left(-e^{-t} + \ln(1+e^{-t})\right) = 1 - \ln 2$$

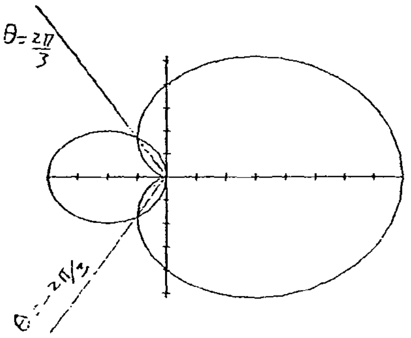
So the integral converges and its value $= 1 - \ln 2$.

13. Q6 $\frac{dx}{dt} = t^4$, $\frac{dy}{dt} = t^3$, $\qquad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^8+t^6} = t^3\sqrt{t^2+1}$

$$\begin{aligned}
\int t^3\sqrt{t^2+1}\,dt &\overset{(t=\tan\theta)}{=} \int \tan^3\theta\sec^3\theta\,d\theta = \int \left(\tan^2\theta\sec^2\theta\right)\tan\theta\sec\theta\,d\theta \overset{(u=\sec\theta,\ \tan^2\theta=\sec^2\theta-1)}{=} \int (u^2-1)u^2\,du \\
&= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(\sec\theta)^5 - \tfrac{1}{3}(\sec\theta)^3 + C = \tfrac{1}{5}\left(t^2+1\right)^{5/2} - \tfrac{1}{3}\left(t^2+1\right)^{3/2} + C \\
&= \tfrac{1}{15}\left(t^2+1\right)^{3/2}\left(3t^2-2\right) + C
\end{aligned}$$

The length $\displaystyle = \int_0^1 t^3\sqrt{t^2+1}\,dt = \frac{1}{15}(2)^{3/2} + \frac{2}{15} = \frac{2\sqrt{2}}{15} + \frac{2}{15}$

14. Q7 The two graphs are shown below. The graphs interesct at the pole and where $0 \le \theta \le 2\pi$ satisfie $1+\cos\theta = -\cos\theta$, or $\cos\theta = -\frac{1}{2}$. Thus $\theta = \pm\frac{2\pi}{3}$.



$$\begin{aligned}
\text{The area} &= \int_{\pi/2}^{2\pi/3} \cos^2\theta\,d\theta + \int_{2\pi/3}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \int_{\pi/2}^{\pi} \cos^2\theta\,d\theta + \int_{2\pi/3}^{\pi} (1+2\cos\theta)\,d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos 2\theta)\,d\theta + [\theta + 2\sin\theta]_{2\pi/3}^{\pi} = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/2}^{\pi} + \frac{\pi}{3} - \sqrt{3} = \frac{7\pi}{12} - \sqrt{3}
\end{aligned}$$